



Interception Conditions in Multi-Agent Pursuit-Evasion Game with Integral and Geometric Constraints on Control Functions

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Abstract

In this paper, the authors carefully examine a pursuit-evasion differential game problem involving multiple number of pursuers and one evader in the Hilbert space l_2 . Players' motions obey certain first-order differential equations. Control functions of the pursuers and evader are subject to integral and geometric constraints respectively. The payoff function is the infimum of the distance between evader and closest pursuer when the game comes to an end at a fixed time θ which is termed as duration of the game. During the course of the game, pursuers try to minimize the payoff and the evader tries to do the opposite. The authors obtain the value of the game and construct optimal strategies of the players. The results are established through the construction of attainability domains and the solution of an auxiliary game involving one pursuer and one evader. Conditions are obtained under which the pursuer can guarantee equality of the states at the terminal time, and these conditions are then extended to the multi-pursuer setting. A positive number ϕ is introduced, defined as the infimum of all shifts l for which the evader's attainability ball is covered by the union of the pursuers' shifted attainability balls. Under a geometric condition on the initial positions, this number is shown to coincide with the value of the game. An illustrative example in l_2 is presented to demonstrate the applicability of the established results.

Keywords: Pursuit–evasion differential games; multi-agent systems; integral control constraints; geometric control constraints.

1. Introduction

A considerable body of research has been devoted to pursuit- evasion differential games in which several pursuers attempt to capture one evader in finite and infinite dimensional spaces. Particular attention has been given to games formulated in Hilbert spaces, such as l_2 , where the player's controls are subject to various constraints. These constraints include integral, geometric, generalized integral, and, more recently, Grunall- type constraints. Among the works that have contributed to the development of this area are [1]- [13], [26]- [28], and the references therein.

In most of the aforementioned studies, the motion of the players is governed by first- order differential equations of the form

$$\begin{cases} P_i: \dot{x}_i = a(t)u_i(t), x_i(0) = x_{i0}, i = 1, 2, \dots, \\ E: \dot{y} = b(t)v(t), y(0) = y_0, \end{cases} \quad (1.1)$$

1 where $u_i(t)$ and $v(t)$ are the control functions of the pursuers and evader, respectively, while $a(t)$ and $b(t)$ are scalar measurable functions satisfying appropriate conditions. Depending on the particular problem under consideration, the control functions are subject to integral or geometric constraints. A major objective in many of these studies is to determine the value of the game and construct optimal strategies for the players. Several studies have considered pursuit- evasion differential games governed by the dynamics in (1.1) under integral and geometric constraints. For example, a differential game involving a finite number of pursuers and a single evader was investigated in [32]. In that study, $a(t) = b(t) = 1$, while the control functions of the evader and some of the pursuers are subject to integral constraints and those of the remaining pursuers are subject to geometric constraints. The payoff is defined as the distance between the evader and the closest pursuer at the termination time of the game. The authors constructed optimal strategies for the players and determined the value of the game. A related problem was considered in [27], where the value of the game was obtained using different optimal strategies for the players.

Ibragimov [12] studied a pursuit- evasion differential game involving many pursuers and one evader in the Hilbert space l_2 . The player's dynamics are described by (1.1) with $a(t) = b(t) = 1$, and the control functions are subject to geometric constraints. Optimal strategies were constructed and the value of the game was obtained. The same class of problems was subsequently investigated under integral constraints in [11] and [23], where similar results were established. The papers [9], [10], and [13] also considered pursuit- evasion differential games governed by (1.1), with $a(t) = b(t) = (\theta - t)$, where θ denotes the duration of the game. Different types of constraints were imposed on the player's control functions in these studies. In each case, optimal strategies were constructed and the value of the game was determined.

Adamu et al. [1] investigated a pursuit- evasion differential game governed by (1.1) in a Hilbert space, with integral constraints imposed on the player's control functions. In their formulation, $a(t) = 1$ and $b(t) = \theta - t$. Under appropriate conditions, the authors determined the value of the game and constructed optimal strategies for the players. In [2], a related game with $a(t) = b(t)$ and geometric constraints on the player's control functions was studied. Conditions for determining the value of the game were established, and optimal strategies were constructed. Ahmed [32] considered a pursuit problem involving a finite number of players on a closed convex subset K of $\mathbb{R}^n$, with generalized integral constraints imposed on the player's control functions. These studies demonstrate the broad range of constraint structures that have been incorporated into pursuit- evasion differential games.

More recently, Grnwall- type constraints have been introduced into pursuit- evasion differential games. Such constraints provide a framework that incorporates not only restrictions related to the instantaneous magnitude or speed of a player's control but also information associated with the accumulated energy of the control. Rilwan et al. [33] studied a fixed duration pursuit- evasion differential game involving one pursuer and one evader, with Grnwall- type constraints imposed on the control functions of all players. The player's dynamics were governed by simple differential equations. The authors established sufficient conditions under which pursuit can be completed and evasion is possible. They also constructed attainability domains and optimal strategies for the players.

In [34], Rilwan et al. investigated the value of a fixed duration pursuit- evasion differential game involving countably many pursuers and one evader under Grnwall- type constraints. The Grnwall- type constraint was considered as a generalization of the classical geometric constraint imposed on the player's control functions. The authors constructed the attainability domain of the players and used the resulting strategies to establish an estimate for the value of the game. Gbande et al. [6] further contributed to this line of research by studying pursuit problems involving multiple players under combined geometric and Grnwall- type constraints. Sufficient conditions for successful pursuit were established. Other related studies have considered different constraint structures and infinite dimensional formulations. Borodo et al. [5] investigated a pursuit differential game in the sequence space l_∞ , in which both the state trajectories and control functions of the players are defined in l_∞ . The model consists of finitely many pursuers and a single evader, with componentwise integral constraints imposed on the player's control functions. The authors established conditions under which permissible pursuer strategies exist that guarantee capture, namely, $x_i(\theta) = y(\theta)$ for some $i \in \{1, 2, \dots, m\}$ at a certain time $\theta > 0$.

In a related study, Badakaya et al. [4] considered pursuit problems in the Hilbert space l_2 involving multiple pursuers and a single evader. The pursuers were governed by second- order dynamics, whereas the evader was described by a first- order differential equation. Sufficient conditions for capture in the l - sense were obtained, and piecewise strategies were constructed for the pursuers. Umar et al. [35] studied a pursuit differential game governed by an infinite system of binary differential equations in the Hilbert space l_2 , with geometric constraints

imposed on the players' control functions. By constructing an optimal pursuer strategy, the authors derived estimates for the guaranteed time required to drive the system state to the origin.

More recently, Ibragimov et al. [7] considered differential games involving multiple pursuers and one evader under particular forms of integral constraints. These results further demonstrate the increasing importance of infinite dimensional and multi- player formulations in the study of pursuit- evasion problems. Although substantial progress has been made in the study of pursuit- evasion differential games, most existing results have focused on relatively simple first- order dynamics of the form (1.1), together with integral, geometric, generalized integral, or Grunall- type constraints considered separately or in selected combinations. Moreover, many of the available sufficient conditions for successful pursuit or evasion are based on comparisons between the maximum speeds or maximum energies of the pursuers and the evader. In this piece of research, we study pursuit- evasion differential game problem described by (1.1) in a Hilbert space l_2 with control functions of the pursuers subject to integral constraints. Whereas, geometric constraint is imposed on the control function of the evader. This problem is missing in the literature and it excludes bases of comparison between maximum energy of the pursuers to that of the evader or comparison between the maximum speeds of the two opposing players. This is because of the different types of constraints on control functions considered for the two groups of players.

2. Problem Formulation

Consider the Hilbert space $l_2 = \{x = (x_1, x_2, \dots) : \sum_{n=1}^{\infty} x_n^2 < \infty\}$, with inner product $\langle \cdot, \cdot \rangle : l_2 \times l_2 \rightarrow R$ and norm $\|\cdot\| : l_2 \rightarrow [0, +\infty)$, defined as follows:

$$\langle x, y \rangle = \sum_{n=1}^{\infty} x_n y_n, \quad \|x\| = \left(\sum_{n=1}^{\infty} x_n^2 \right)^{1/2},$$

where $x, y \in l_2$

In what will follow we shall denote a ball and sphere each centered at a and with radius r by $B_r(a)$ and $S_r(a)$ respectively. That is $B_r(a) = \{x \in l_2 : \|x - a\| \leq r\}$ and $S_r(a) = \{x \in l_2 : \|x - a\| = r\}$.

Also we let $C(0, \theta, l_2)$ be the space of functions $g(t) = (g_1(t), g_2(t), \dots, g_j(t), \dots) \in l_2$ $t \geq 0$, such that the following conditions hold:

(a) $g_j(t)$, $0 \leq t \leq \theta$, $j = 1, 2, \dots$, are absolutely continuous functions;

(b) $g(t)$, $0 \leq t \leq \theta$, is a continuous function in the norm of l_2

Let the players move according to the following first- order differential equations:

$$\begin{cases} P_i: \dot{x}_i = a(t)u_i(t), x_i(0) = x_{i0}, i \in I, \\ E: \dot{y} = b(t)v(t), y(0) = y_0, \end{cases} \quad (2.1)$$

where $P_i, i \in I = \{1, 2, \dots\}$ is the i^{th} pursuer; E is the evader; $x_i, x_{i0}, u_i, y, y_0, v \in l_2$ $u_i = (u_{i1}, u_{i2}, \dots)$ is a control parameter of the i^{th} pursuer and $v = (v_1, v_2, \dots)$ is that of the evader E . Additionally, $a(t)$ and $b(t)$ are positive scalar measurable functions on $[0, \theta]$ that satisfied the following conditions: $b(t) \leq a(t)$, $\forall t \in [0, \theta]$ and

$$A = \left(\int_0^\theta a^2(t) dt \right)^{\frac{1}{2}} < \infty, B = \left(\int_0^\theta b^2(t) dt \right)^{\frac{1}{2}} < \infty.$$

Definition 2.1. A function $u_i(\cdot) : [0, \theta] \rightarrow l_2$ with Borel measurable coordinates u_{ik} $k = 1, 2, \dots$, such that

$$\int_0^\theta \|u_i(t)\|^2 dt \leq \beta_i^2, \quad (2.2)$$

where β_i is a positive number, is referred to as admissible control of the i^{th} pursuer subject to integral constraints.

Definition 2.2. A function $v(\cdot) : [0, \theta] \rightarrow l_2$ with Borel measurable coordinates $v_k(t)$ $k = 1, 2, \dots$, such that

$$\|v(t)\| \leq \gamma, \quad t \geq 0 \quad (2.3)$$

where γ is a positive number, is referred to as admissible control of the evader E subject to geometric constraints. If the pursuers P_i and evader E choose their admissible controls $u_i(\cdot)$ and $v(\cdot)$ respectively, the solutions to the equations in (2.1) are given by:

$$\begin{aligned}x_i(\theta) &= x_{i0} + \int_0^\theta a(s)u_i(s)ds, \\y(\theta) &= y_0 + \int_0^\theta b(s)v(s)ds.\end{aligned}\quad (2.5)$$

One can readily see that $x_i(\cdot), y(\cdot) \in C(0, \theta, l_2)$

Definition 2.3. The payoff of the game is the infimum of the distances between the evader and the pursuers at θ and is given by

$$\Psi(u_i, v) = \inf_{i \in I} \|x_i(\theta) - y(\theta)\|.$$

The target of the pursuer is to minimize the payoff, and that of the evader is to maximize it.

Definition 2.4. A strategy of the i^{th} pursuer is a function $U_i(t, x_i, y, v(t)) : [0, \theta] \times l_2 \times l_2 \times l_2 \rightarrow l_2$, such that the system

$$\begin{cases} \dot{x}_i = a(t)U_i(t, x_i, y, v(t)), & x_i(0) = x_{i0}, \\ \dot{y} = b(t)v(t), & y(0) = y_0 \end{cases} \quad (2.6)$$

has a unique solution $(x_i(\cdot), y(\cdot))$ with $x_i(\cdot), y(\cdot) \in C(0, \theta, l_2)$, for arbitrary admissible control $v(\cdot)$ of the evader E . A strategy U_i is said to be admissible if each control formed by the strategy $U_i(t, x_i(t), y(t), v(t))$ is admissible.

Definition 2.5. A strategy of the evader is a function $V(t, x_i, \dots, x_m, \dots, y) : [0, \infty) \times l_2 \times \dots \times l_2 \times \dots \times l_2 \rightarrow l_2$, such that the system

$$\begin{cases} \dot{x}_i = a(t)u_i, & x_i(0) = x_{i0}, \quad i \in I \\ \dot{y} = b(t)V(t, x_1, \dots, x_m, \dots, y), & y(0) = y_0, \end{cases} \quad (2.7)$$

has a unique solution $(x_i(\cdot), y(\cdot))$ and that $x_i(\cdot), y(\cdot) \in C(0, \theta, l_2)$, for arbitrary admissible controls $u_i = u_i(t)$, $0 \leq t \leq \theta$, of the pursuers P_i . A strategy V of the evader is said to be admissible if each control formed by this strategy is admissible.

Definition 2.6. Optimal strategies of the pursuers are strategies $\bar{U}_i$, $i \in I$ such that

$$\inf_{u_i, i \in I} \Phi_1(U_1, U_2, \dots) = \Phi_1(\bar{U}_1, \bar{U}_2, \dots),$$

where $\Phi_1(U_1, U_2, \dots) = \sup_{v(\cdot)} \inf_{i \in I} \|x_i(\theta) - y(\theta)\|$; $U_i(\cdot)$, $i \in I$ are admissible strategies of the pursuers and $v(\cdot)$ is an admissible control of the evader.

Definition 2.7. The strategy $\bar{V}$ of the evader is said to be optimal if

$$\sup_V \Phi_2(V) = \Phi_2(\bar{V}),$$

where $\Phi_2(V) = \inf_{u_i(\cdot), i \in I} \inf_{i \in I} \|x_i(\theta) - y(\theta)\|$, $u_i(\cdot)$ are admissible controls of the pursuers P_i and $V(\cdot)$ is an admissible strategy of the evader.

It is well known that the game (2.1)- (2.3) has value Δ if $\Phi_1(\bar{U}_i) = \Delta = \Phi_2(\bar{V})$ (see, for example, [24]).

Research question: Find the optimal strategies $\bar{U}_i$, and $\bar{V}$ of the pursuers P_i , $i \in I$ and E respectively, and the value of the game (2.1)- (2.3).

3. Attainability Domains

Proposition 3.1. The attainability domain of the

(1) pursuer P_i from the initial position x_{i0} at time $t = 0$ until the time θ is the ball with center at x_{i0} and radius $R_i = A\beta_i$. That is $B_{R_i}(x_{i0}) = \{x \in l_2 : \|x - x_{i0}\| \leq A\beta_i\}$.

(2) evader E from the initial position y_0 at time $t = 0$ until the time θ is the ball with center at y_0 and radius $r = B\gamma\theta$. That is $B_r(y_0) = \{x \in l_2 : \|x - y_0\| \leq B\gamma\theta\}$.

Proof For the proof of (1), we have

$$\begin{aligned}
\|x_i(\theta) - x_{i0}\| &= \left\| x_{i0} + \int_0^\theta a(s)u_i(s)ds - x_{i0} \right\| \\
&= \left\| \int_0^\theta a(s)u_i(s)ds \right\| \leq \int_0^\theta \|a(s)u_i(s)\| ds \quad (3.1) \\
&\leq \left(\int_0^\theta a^2(s)ds \right)^{\frac{1}{2}} \left(\int_0^\theta \|u_i(s)\|^2 ds \right)^{\frac{1}{2}} \\
&\leq A\beta_i,
\end{aligned}$$

(here we used Cauchy- Schwartz inequality).

Let $\bar{x} \in B_{R_i}(x_{i0})$. If the pursuer P_i uses the control $u_i(s) = \frac{a(s)(\bar{x}-x_{i0})}{A^2}$ $0 \leq s \leq \theta$, then we have

$$x_i(\theta) = x_{i0} + \int_0^\theta a(s)u_i(s)ds = x_{i0} + \int_0^\theta \frac{a^2(s)(\bar{x}-x_{i0})}{A^2}ds = x_{i0} + \frac{\bar{x}-x_{i0}}{A^2} \int_0^\theta a^2(s)ds = x_{i0} + \bar{x} - x_{i0} = \bar{x}.$$

This proves (1). The proof of (2) is similar, with evader's control $v(t) = \frac{b(t)}{B^2}(\bar{y} - y_0)$. This complete the proof of the proposition.

4.Auxiliary Game

In this section, we consider differential game problem with one pursuer and one evader (auxiliary game) described by the following equations:

$$\begin{cases} P: \dot{x}_1 = a(t)u_1(t), x_1(0) = x_{10} \\ E: \dot{y} = b(t)v(t), y(0) = y_0. \end{cases} \quad (4.1)$$

We define the half space

$$\Omega = \{z \in l_2: 2\langle y_0, -x_{10}z \rangle \leq A^2(\beta_1^2 - \gamma^2\theta) + \|y_0\|^2 - \|x_0\|^2\},$$

where β_1 is the maximum energy of the only one pursuer in the auxiliary game. The goal of the pursuer P_1 is to ensure the equality $x_1(\theta) = y(\theta)$ and that of evader E is the opposite. The problem is to construct admissible strategy for the pursuer that ensure the equality $x_1(\theta) = y(\theta)$ for any admissible control of the evader.

Lemma 4.1. If $y(\theta) \in \Omega$ then there exist a strategy of the pursuer such that $x_1(\theta) = y(\theta)$ in the auxiliary game. 4.2) is admissible. Indeed, using polarization identity, we have

$$\begin{aligned}
\int_0^\theta \|u_1(s)\|^2 ds &= \int_0^\theta \left\| \frac{a(s)(y_0 - x_{10})}{A^2} + \frac{b(s)}{a(s)}v(s) \right\|^2 ds \\
&= \int_0^\theta \frac{\|y_0 - x_{10}\|^2}{A^4} a^2(s)ds + \int_0^\theta 2 \left\langle \frac{a(s)(y_0 - x_{10})}{A^2}, \frac{b(s)}{a(s)}v(s) \right\rangle ds \quad (4.3) \\
&+ \int_0^\theta \frac{b^2(s)}{a^2(s)} \|v(s)\|^2 ds
\end{aligned}$$

The first term on the right-hand side of (4.3) yields

$$\begin{aligned}
\int_0^\theta \frac{\|y_0 - x_{10}\|^2}{A^4} a^2(s)ds &= \frac{\|y_0 - x_{10}\|^2}{A^4} \int_0^\theta a^2(s)ds \\
&= \frac{\|y_0 - x_{10}\|^2}{A^4} A^2 = \frac{\|y_0 - x_{10}\|^2}{A^2} \quad (4.4)
\end{aligned}$$

For the second term, the inclusion $y(\theta) \in \Omega$ implies that

$$2\langle y_0, -x_{10}y(\theta) \rangle \leq A^2(\beta_1^2 - \gamma^2\theta) + \|y_0\|^2 - \|x_{10}\|^2. \quad (4.5)$$

In accordance with the inequality (4.5) and using the state equation of the evader (2.5), we have

$$\begin{aligned}
\int_0^\theta 2 \left\langle \frac{a(s)(y_0 - x_{10})}{A^2}, \frac{b(s)}{a(s)} v(s) \right\rangle ds &= \int_0^\theta \frac{2}{A^2} \langle y_0, -x_{10} b(s) v(s) \rangle ds \\
&= \frac{2}{A^2} \left\langle y_0, -x_{10} \int_0^\theta b(s) v(s) ds \right\rangle \\
&= \frac{2}{A^2} \langle y_0, -x_{10} y(\theta) - y_0 \rangle \\
&= \frac{1}{A^2} (2 \langle y_0, -x_{10} y(\theta) \rangle - 2 \langle y_0, -x_{10} y_0 \rangle) \quad (4.6) \\
&= \frac{1}{A^2} (2 \langle y_0, -x_{10} y(\theta) \rangle - 2 \|y_0\|^2 + 2 \langle x_{10}, y_0 \rangle) \\
&\leq \frac{1}{A^2} (A^2 (\beta_1^2 - \gamma^2 \theta) - \|y_0 - x_{10}\|^2) \\
&\leq \beta_1^2 - \gamma^2 \theta - \frac{\|y_0 - x_{10}\|^2}{A^2}.
\end{aligned}$$

The third term in (4.3) yields

$$\int_0^\theta \frac{b^2(s)}{a^2(s)} \|v(s)\|^2 ds \leq \int_0^\theta \|v(s)\|^2 ds \leq \gamma^2 \theta. \quad (4.7)$$

Clearly, using the inequalities (4.4), (4.6) and (4.7) in equation (4.3), we obtained

$$\int_0^\theta \|u(t)\|^2 dt \leq \beta_1^2 \quad (4.8)$$

Lastly, if the pursuer P_1 uses the strategy (4.2) then we have $x_1(\theta) = y(\theta)$. Indeed,

$$\begin{aligned}
x_1(\theta) &= x_0 + \int_0^\theta a(s) \left(\frac{a(s)(y_0 - x_{10})}{A^2} + \frac{b(s)}{a(s)} v(s) \right) ds = x_{10} + \int_0^\theta \left(\frac{y_0 - x_{10}}{A^2} \right) a^2(s) ds + \int_0^\theta b(s) v(s) ds = x_{10} + \\
&y_0 - x_{10} + \int_0^\theta b(s) v(s) ds = y(\theta).
\end{aligned}$$

5. Main Result

In this section, we present the main result of the paper. That is we construct optimal strategies of the players and find the value of the game. Firstly, we present some useful lemmas in the presentation of the main result.

Lemma 5.1. (see [9], Assertion 4) If there exists a nonzero vector p_0 such that $\langle y_0, -x_{i0} p_0 \rangle \geq 0$, for all $i \in I$, and $B_r(y_0) \subset \bigcup_{i \in I} B_{R_i}(x_{i0})$ then $B_r(y_0) \subset \bigcup_{i \in I} X_i$, where;

Lemma 5.2. (see [9], Assertion 5) Let $\inf_{i \in I} R_i = R_0 > 0$, If there exists a nonzero vector p_0 such that $\langle y_0, -x_{i0} p_0 \rangle \geq 0$ for all $i \in I$ and for any $\epsilon > 0$ the set $\bigcup_{i \in I} B_{R_i - \epsilon}(x_{i0})$ does not contain the ball $B_r(y_0)$, then there exist a point $\bar{y} \in S_r(y_0)$ such that $\|\bar{y} - x_{i0}\| \geq R_i$ for all $i \in I$.

We define a positive number ϕ by

$$\phi = \inf \left\{ l \geq 0 : B_r(y_0) \subset \bigcup_{i \in I} B_{R_i + l}(x_{i0}) \right\}, \quad (5.1)$$

where $R_i = A\beta_i$ and $r = B\gamma\theta$.

Theorem 5.1. If there exists a nonzero vector p_0 such that $\langle y_0, -x_{i0} p_0 \rangle \geq 0$, for all $i \in I$, then the number ϕ defined by (5.1) is the value of the game (2.1)- (2.3).

Proof To prove this theorem, we first introduce dummy pursuers with state variables z_i , $i \in I$ and motions described by the equations

$$\dot{z}_i = w_i(t), z_i(0) = x_{i0},$$

where the control function $w_i(t)$ is such that

$$\left(\int_0^\theta \|w_i(s)\|^2 ds \right)^{\frac{1}{2}} \leq \bar{\beta}_i = \beta_i + \frac{\phi}{A}.$$

Clearly, $\bar{\beta}_i > \beta_i$ and $(\bar{\beta}_i - \beta_i)A = \phi$. Let strategies of the dummy pursuers z_i , $i \in I$ be defined by

$$w_i(t) = \begin{cases} \frac{a(t)(y_0 - x_{i0})}{A^2} + \frac{b(t)}{a(t)}v(t), & 0 \leq t \leq \tau, \\ 0, & \tau < t \leq \theta, \end{cases} \quad (5.2)$$

where τ is the time such that

$$\int_0^\tau \left| \frac{a(t)(y_0 - x_{i0})}{A^2} + \frac{b(t)}{a(t)}v(t) \right|^2 dt = \bar{\beta}_i.$$

Using the strategy of the dummy pursuer, we define the strategies of the real pursuers P_i as follows:

$$u_i(t) = \frac{\beta_i}{\bar{\beta}_i} w_i(t). \quad (5.3)$$

In accordance with the payoff of the game $\Psi(u, v) = \inf_{i \in I} \|x_i(\theta) - y(\theta)\|$, the number ϕ defined in (5.1) is the value of the game if the following inequalities are satisfied:

(i) $\sup_{v(\cdot)} \Psi(u_i, v) \leq \phi$, (ii) $\inf_{u_i(\cdot)} \Psi(u_i, v) \geq \phi$,

where $v(\cdot)$ and $u_i(\cdot)$ are arbitrary admissible controls of the evader E and pursuers P_i respectively.

To prove the inequality (i), by definition of ϕ , we have $B_r(y_0) \subset \bigcup_{i=1}^\infty B_{R_i}(x_{i0})$. Then using lemma (5.1), we have $B_r(y_0) \subset \bigcup_{i=1}^\infty X_i$, where

Consequently, the point $y(\theta) \in B_r(y_0)$ belong to some half space X_s , $s \in I$. By lemma (4.1) for strategy (5.2) of the dummy pursuers z_i , we obtain $z_s(\theta) = y(\theta)$. By taking account of (5.3), we have

$$\begin{aligned} \|y(\theta) - x_s(\theta)\| &= \|z_s(\theta) - x_s(\theta)\| \\ &= \left\| x_{i0} + \int_0^\theta a(t)w_s(t)dt - x_{i0} - \int_0^\theta a(t)u_s(t)dt \right\| \\ &= \left\| \int_0^\theta a(s)w_s(t)dt - \int_0^\theta \frac{\beta_s}{\bar{\beta}_s} a(s)w_s(t)dt \right\| \quad (5.4) \\ &\leq \int_0^\theta \left| \left(1 - \frac{\beta_s}{\bar{\beta}_s}\right) a(s)w_s(t) \right| dt \\ &\leq \left(\frac{\beta_s - \bar{\beta}_s}{\bar{\beta}_s} \right) \int_0^\theta \|a(s)w_s(t)\| dt \end{aligned}$$

Using Cauchy- Schwartz inequality, we have

$$\int_0^\theta \|a(s)w_s(t)\| dt \leq \left(\int_0^\theta a^2(t)dt \right)^{\frac{1}{2}} \left(\int_0^\theta \|w_s(t)\|^2 dt \right)^{\frac{1}{2}} \leq A\bar{\beta}_s \quad (5.5)$$

Therefore, from inequality (5.4) and (5.5), we have $\|y(\theta) - x_s(\theta)\| \leq \phi$. This prove the inequality (i).

To prove the inequality (ii). If $\phi = 0$, the inequality follows for any admissible control of evader. Suppose that $\phi > 0$ and let the evader uses the strategy

$$v(t) = \frac{b(t)}{B^2} (\bar{y} - y_0), 0 \leq t \leq \theta. \quad (5.6)$$

By the definition of ϕ , and for any $0 < \epsilon < \phi$, the ball $B_r(y_0)$ is not contained in the set $\bigcup_{i=1}^\infty B_{R_i}(x_{i0})$. Then by lemma (5.2), there exist a point $\bar{y} \in S_r(y_0)$, such that

$$\|\bar{y} - x_{i0}\| \geq R_i. \quad (5.7)$$

In accordance with the inequalities (3.1) and (5.7), we have for all $i \in I$

$$\|\bar{y} - x_i(\theta)\| \geq \|\bar{y} - x_{i0}\| - \|x_{i0} - x_i(\theta)\| \geq A\beta_i + \phi - A\beta_i = \phi.$$

The control (5.6) brings the evader to the point $\bar{y}$ at the time θ . This claim can be proved as follows:

$$y(\theta) = y_0 + \int_0^\theta b(s)v(s)ds = y_0 + \int_0^\theta \frac{b^2(s)}{B^2} (\bar{y} - y_0)ds = y_0 + \frac{1}{B^2} (\bar{y} - y_0) \int_0^\theta b^2(s)ds = y_0 + \bar{y} - y_0 = \bar{y}.$$

Thus, $\inf_{i \in I} \|x_i(\theta) - y(\theta)\| \geq \phi$. This proves the inequality (ii). Hence, the proof the theorem is complete.

6. Example

We consider a game problem in which the players dynamics equations are given by where the controls $u_i(t)$ and $v(t)$, $t \in [0, \theta]$ satisfy (2.2) and (2.3) respectively. Clearly, $b(t) = e^{-t} \leq e^t = a(t)$ for all t . Let $y_0 = \mathbf{0} = (0, 0, 0, \dots)$ and $x_{i0} = (0, 0, \dots, \sqrt{13}, 0, \dots)$ where the number $\sqrt{13}$ is in the i^{th} coordinate. Suppose that $\theta = \ln 2$ and the maximum energy of the i^{th} pursuer is $\beta_i = \sqrt{6}$ for all $i \in I$. Also let the maximum speed of the evader be given by $\gamma = \frac{2\sqrt{2}}{\ln 2}$. In relation to these assumptions, observe that $R_i = 3$; $r = \sqrt{3}$.

We now claim the following:

(1) $B_{\sqrt{3}}(\mathbf{0}) \subset \bigcup_{i \in I} B_{4+l}(x_{i0})$ holds for any $l > 0$ (2) The set $\bigcup_{i \in I} B_4(x_{i0})$ does not contain the ball $B_{\sqrt{3}}(\mathbf{0})$.

In view of these claims, then according to the Theorem 5.1 the value of the game (6.1) is given by $\phi = 1$. With this, it remains to prove the two claims.

To prove the first claim, let $\tilde{y} \in B_{\sqrt{3}}(\mathbf{0})$ then we have $\sum_{i \in I} \tilde{y}_i^2 \leq 3$. Then the vector $\tilde{y}$ has a nonnegative coordinate or all the coordinates are negative.

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$$\begin{aligned} \|\tilde{y} - x_{k0}\| &= \left((\sqrt{13} - \tilde{y}_k)^2 + \sum_{r=1}^{k-1} \tilde{y}_r^2 + \sum_{r=k+1}^{\infty} \tilde{y}_r^2 \right)^{\frac{1}{2}} \\ &= \left(13 - 2\tilde{y}_k + \sum_{r=1}^{\infty} \tilde{y}_r^2 \right)^{\frac{1}{2}} \\ &\leq (13 - 2\tilde{y}_k + 3)^{\frac{1}{2}} \leq 4 < 4 + l. \end{aligned} \quad (6.1)$$

This means that $\tilde{y} \in B_{4+l}(x_{k0})$. Moreover, if all the coordinates of $\tilde{y}$ are negatives then for sufficiently large k , the inequality (6.1) holds. This is because, by the convergence of the series $\sum_{r=1}^{\infty} \tilde{y}_r^2$, we have $\tilde{y}_k$ approaches zero as k approaches infinity. In the other hand, any point $\tilde{y} \in S_{\sqrt{3}}(\mathbf{0})$ whose coordinates are negatives does not belong to the set $\bigcup_{i \in I} B_4(x_{i0})$. This is because $\|\tilde{y} - x_{i0}\| = (16 - 2\tilde{y}_i)^{\frac{1}{2}} > 4$ for all $i \in I$. Hence the value of the game is the number

$$\phi = \left\{ l \geq 0 : B_{\sqrt{3}}(\mathbf{0}) \subset \bigcup_{i \in I} B_{4+l}(x_{i0}) \right\} = 1.$$

7. Conclusion

We have studied pursuit-evasion differential game in which a multiple number pursuers chased one evader in a Hilbert space l_2 . Control function of each pursuer and evader are subject to integral and geometric constraints respectively. We found value of the game under certain conditions and constructed optimal strategies of the players. The result presented comprises solution of an auxiliary game and a theorem. The auxiliary game involves one pursuer and evader in which we proved that it is possible the pursuer to achieve the equality $x_1(\theta) = y(\theta)$. This is shown when the pursuer adapts a strategy different from the strategy of parallel approach. The theorem is the main result which provides sufficient conditions for finding value of the game.

References

- Adamu, J., Muanghoo, K., Badakaya, A. J., & Rilwan, J. (2020). On pursuit-evasion differential game problem in a Hilbert space. *AIMS Mathematics*, 5(6), 7467–7479.
- Ahmad, I., Kumam, P., Ibragimov, G. I., Rilwan, J., & Kumam, W. (2019). An optimal pursuit differential game problem with one evader and many pursuers. *Mathematics*, 7(9), 842.
- Ahmed, I., Wiyada, K. W., Ibragimov, G. I., & Rilwan, J. (2020). Pursuit differential game problem with multiple players on a closed convex set with more general integral constraints. *Thai Journal of Mathematics*, 18(2), 551–561. <http://thaijmath.in.cmu.ac.th>
- Badakaya, A. J. (2017). Value of differential game problems with multiple players in a certain Hilbert space. *Journal of the Nigerian Mathematical Society*, 8, 287–305.

- Badakaya, A. J., Tsoho, S. M., & Salmi, M. (2024). On some l -catch pursuit differential games with different players. *Dynamic Equations, Differential Equations and Dynamical Systems*. <https://doi.org/10.1007/s12591-023-00673-8>
- Borodo, B. B., Minjibir, M. S., & Badakaya, A. J. (2025). Pursuit differential game problem with component-wise constraints in l_∞ . *Journal of Optimization Theory and Applications*, 207, Article 31. <https://doi.org/10.1007/s10957-025-02785-3>
- Gbande, D. T., Cheme, D. T., Nyimkor, B., & Adamu, R. A. (2025). Pursuit problem of a linear differential game for many numbers of players with geometric and Gronwall type constraints on players controls. *Journal of Advances in Mathematics and Computer Science*, 40(4), 57–66. <https://doi.org/10.9734/jamcs/2025/v40i41988>
- Ibragimov, G. I. (2005). Optimal pursuit with countably many pursuers and one evader. *Differential Equations*, 41, 627–635.
- Ibragimov, G. I. (1998). A game of optimal pursuit of one object by several. *Journal of Applied Mathematics and Mechanics*, 62(2), 187–192.
- Ibragimov, G. I., & Salimi, M. (2009). Pursuit-evasion differential game with many inertial players. *Mathematical Problems in Engineering*. <https://doi.org/10.1155/2009/653723>
- Ibragimov, G. I., & Kuchkarov, A. Sh. (2017). Fixed duration pursuit-evasion differential game with integral constraints. *Journal of Physics: Conference Series*, 435. <https://doi.org/10.1088/1742-6596/435/1/012017>
- Ibragimov, G. I., Abd Rashid, N., Kuchkarov, A., & Ismail, F. (2015). Multi-pursuer differential game of optimal approach with integral constraints on controls of the players. *Taiwanese Journal of Mathematics*, 19, 963–976.
- Ibragimov, G. I., Hussin, N. A. (2010). A pursuit-evasion differential game with many pursuers and one evader. *Malaysian Journal of Mathematical Sciences*, 4, 183–194.
- Ibragimov, G. I., Inomiddinov, S., Kurbonbekova, O., Pansera, B. A., & Khodjimmet, A. (2026). Differential game of multiple pursuers and one evader with integral constraint of special form. *Journal of Dynamics and Games*, 14, 72–82. <https://doi.org/10.3934/jdg.2025023>
- Ibragimov, G. I. (2015). Simple motion pursuit differential game of many pursuers and one evader on convex compact set. *International Journal of Pure and Applied Mathematics*, 102(4), 733–745.
- Ivanov, R. P., & Ledyayev, Yu. S. (1981). Optimality of pursuit time in a simple motion differential game of many objects. *Trudy Matematicheskogo Instituta imeni V. A. Steklova*, 158, 87–97.
- Jaafaru, A. B., & Gafurjan, I. (2011). Differential game described by infinite system of DE's of 2nd order: The case of negative coefficients. *International Journal of Pure and Applied Mathematics*, 70(7), 927–938.
- Jaafaru, A. B., & Gafurjan, I. (2012). On some pursuit and evasion differential game problems for an infinite number of first-order differential equations. *Journal of Applied Mathematics*. <https://doi.org/10.1155/2012/717124>
- John, L., Salmon, J. L., Willey, L. C., Casbeer, D., Garcia, E., & Von Mall, A. (2020). Single pursuer and two cooperative evaders in the border defense differential game. *Journal of Aerospace Information Systems*. <https://doi.org/10.2514/1.1010766>
- Leong, W. J., & Ibragimov, G. I. (2008). A multiperson pursuit problem on a closed convex set in Hilbert spaces. *Far East Journal of Applied Mathematics*, 33(2), 205–214.
- Makkapati, V. R., & Tsiotras, P. (2019). Optimal evading strategies and task allocation in multi-player pursuit-evasion problems. *Dynamic Games and Applications*. <https://doi.org/10.1007/s13235-019-00319-x>

- Rilwan, J., Kumam, P., Badakaya, A. J., & Amad, I. (2020). A modified dynamic equation of evasion differential game problem in Hilbert space. *Thai Journal of Mathematics*, 18(1), 199–211.
- Rilwan, J., Ferrara, M., & Badakaya, A. J. (2023). On pursuit and evasion game problems with Gronwall-type constraints. *Quality & Quantity*, 57, 5551–5562. <https://doi.org/10.1007/s11135-023-01616-9>
- Rilwan, J., Fotia, P., & Ferrara, M. (2023). On game value of a differential game problem with Gronwall-type constraints on players control functions. *Decisions in Economics and Finance*, 46, 319–333. <https://doi.org/10.1007/s10203-023-00389-y>
- Salimi, M., Ferrara, M. (2018). Differential game of optimal pursuit of one evader by many pursuers. *International Journal of Game Theory*. <https://doi.org/10.1007/s00182-018-0638-6>
- Salimi, M., Ibragimov, G. I., Stefen, S., & Somayeh, S. (2015). On a fixed duration pursuit differential game with geometric and integral constraints. *Dynamic Games and Applications*. <https://doi.org/10.1007/s13235-015-0161-3>
- Samatov, B. T. (2013). Problems of group pursuit with integral constraints on controls of the players II. *Cybernetics and Systems Analysis*, 49(6), 907–921.
- Satimov, N. Y. (1981). Problem of pursuit and evasion in differential games. *Mathematical Notes*, 29(3), 455–477.
- Soravia, P. (1993). Pursuit-evasion problem and viscosity solution of Isaacs equation. *SIAM Journal on Control and Optimization*, 31, 604–623.
- Subbotin, A. I., & Chentsov, A. G. (1981). *Optimization of guaranteed result in control problem*. Moscow, Russia.
- Sun, W., & Tsiotras, P. (2017). Multiple-pursuer/one-evader pursuit-evasion game in dynamic flowfields. *Journal of Guidance, Control, and Dynamics*, 40. <https://doi.org/10.2514/1.G002125>
- Umar, B. M., Rilwan, J., Danyaro, L. M., Muhammad, A. B., & Bagare, S. A. (2025). Guaranteed pursuit time for an infinite system in l_2 with geometric constraints. *Bangmod International Journal of Mathematics and Computer Science*, 11(4), 63–74. <https://doi.org/10.58715/bangmodjmcs>
- Vagin, D. A., & Petrov, N. N. (2002). A problem of group pursuit with phase constraints. *Journal of Applied Mathematics and Mechanics*, 66, 225–232.
- Von Mall, A., Garcia, E., Casbeer, A., Manickam, S., & Swar, S. C. (2019). Multiple pursuer single evader border defense differential game. *Journal of Aerospace Information Systems*. <https://doi.org/10.2514/6.2019-1162>
- Weintraub, I. T., Patcher, M., & Garcia, E. (2020). *An introduction to pursuit-evasion differential games*. arXiv:2003.05013v1 [math.OC].

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